COLORED MORTON-FRANKS-WILLIAMS INEQUALITIES

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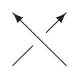
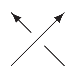
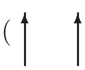
Abstract. We generalize the Morton-Franks-Williams inequality [3, 14] to the colored $\mathfrak{sl}(N)$ link homology defined in [30], which gives infinitely many new bounds for the braid index and the self linking number. A key ingredient of our proof is a composition product for the general MOY graph polynomial, which generalizes that of Wagner [27].

1. Introduction

1.1. **The Morton-Franks-Williams inequality.** The HOMFLY-PT polynomial [4, 21] is an invariant for oriented links in S^3 in the form of a two variable polynomial P. In this paper, we use the following normalization of the HOMFLY-PT polynomial:

$$\begin{cases} x\mathsf{P}(\quad) - x^{-1}\mathsf{P}(\quad) = y\mathsf{P}(\quad), \\ \mathsf{P}(\text{unknot}) = \frac{x-x^{-1}}{y}. \end{cases}$$

Morton [14] and independently, Franks, Williams [3] established the Morton-Franks-Williams inequality, which states that, for a closed braid B with writhe w and b strands,

$$w - b \leq \min \deg_x \mathsf{P}(B) \leq \max \deg_x \mathsf{P}(B) \leq w + b. \tag{1.1}$$

If we consider the $\mathfrak{sl}(N)$ HOMFLY-PT polynomial

$$\mathsf{P}_N = \mathsf{P}|_{x=q^N,\ y=q-q^{-1}},$$

then the Morton-Franks-Williams inequality (1.1) can be expressed as

$$w - b \leq \lim_{N\to\infty} \frac{\min \deg_q \mathsf{P}_N(B)}{N-1} \leq \lim_{N\to\infty} \frac{\max \deg_q \mathsf{P}_N(B)}{N-1} \leq w + b. \tag{1.2}$$

Khovanov and Rozansky [10] categorified the $\mathfrak{sl}(N)$ HOMFLY-PT polynomial. That is, they constructed an invariant $\mathbb{Z}^{\oplus 2}$-graded homology $H_N^{i,j}$ for oriented links such that, for any oriented link L, the graded Euler characteristic of this homology is

$$\sum_{i,j} (-1)^i q^j \dim H_N^{i,j}(L) = \mathsf{P}_N(L).$$

We call i the homological grading of H_N and j the quantum grading of H_N. Dunfield, Gukov, Rasmussen [1] and independently, myself [29] refined (1.2) to

$$w - b \leq \liminf_{N\to\infty} \frac{\min \deg_q H_N(B)}{N-1} \leq \limsup_{N\to\infty} \frac{\max \deg_q H_N(B)}{N-1} \leq w + b, \tag{1.3}$$

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where $\min\deg_q H_N(B)$ and $\max\deg_q H_N(B)$ are the minimal and maximal non-vanishing quantum degrees of $H_N(B)$.

Recall that $w-b$ is the self linking number of the braid B. So the above inequalities provide upper bounds for the maximal self linking number of a given link. It is easy to see that these inequalities also provide lower bounds for the braid index of a given link. For more detailed reviews about these and related inequalities, please see [2, 19].

1.2. **Colored Morton-Franks-Williams inequalities.** In [30], I generalized Khovanov and Rozansky's construction to categorify the Reshetikhin-Turaev $\mathfrak{sl}(N)$ polynomial for links colored by wedge powers of the defining representation of $\mathfrak{sl}(N;\mathbb{C})$. That is, I constructed an invariant $\mathbb{Z}^{\oplus 2}$-graded homology $H_N^{i,j}$ for oriented links colored by wedge powers of the defining representation of $\mathfrak{sl}(N;\mathbb{C})$ such that, for any such colored oriented link L, the graded Euler characteristic of this homology is

$$\sum_{i,j}(-1)^i q^j \dim H_N^{i,j}(L) = \mathsf{P}_N(L),$$

where $\mathsf{P}_N(L)$ is the Reshetikhin-Turaev $\mathfrak{sl}(N)$ polynomial of L. Again, we call i the homological grading of H_N and j the quantum grading of H_N. Moreover, for simplicity, instead of saying something is colored by the m-fold wedge power of the defining representation of $\mathfrak{sl}(N;\mathbb{C})$, we simply say that it is colored by m. If all components of a link are colored by 1, then we say that the link is uncolored. The Reshetikhin-Turaev $\mathfrak{sl}(N)$ polynomial of an uncolored link is the $\mathfrak{sl}(N)$ HOMFLY-PT polynomial of this link. The colored $\mathfrak{sl}(N)$ link homology of an uncolored link is isomorphic to its Khovanov-Rozansky $\mathfrak{sl}(N)$ homology.

The goal of the present paper is to show that the Morton-Franks-Williams inequality generalizes to the colored $\mathfrak{sl}(N)$ link homology. We do so by establishing the following technical result.

Proposition 1.1. *Let B be a closed braid with b strands, l_+ positive crossings and l_- negative crossings. Denote by $l = l_+ + l_-$ the total number of crossings in B. Recall that the writhe of B is $w = l_+ - l_-$. For a positive integer m, denote by $B^{(m)}$ the colored link obtained by coloring B entirely by m. Then, for any $N > m$,*

$$w-b+\frac{w-ml}{N-m} \leq \frac{\min\deg_q H_N(B^{(m)})}{m(N-m)} \leq \frac{\max\deg_q H_N(B^{(m)})}{m(N-m)} \leq w+b+\frac{w+ml}{N-m}, \tag{1.4}$$

where $\min\deg_q H_N(B^{(m)})$ and $\max\deg_q H_N(B^{(m)})$ are the minimal and maximal non-vanishing quantum degrees of $H_N(B^{(m)})$.

Letting $N\to\infty$ in (1.4), we easily get the following colored homological Morton-Franks-Williams inequalities.

Theorem 1.2. *Let B be a closed braid with writhe w and b strands. Then*

$$w-b \leq \liminf_{N\to\infty}\frac{\min\deg_q H_N(B^{(m)})}{m(N-m)} \leq \limsup_{N\to\infty}\frac{\max\deg_q H_N(B^{(m)})}{m(N-m)} \leq w+b. \tag{1.5}$$

More generally, for any two sequences $\{m_k\}$ *and* $\{N_k\}$ *of positive integers satisfying* $\lim_{k\to\infty}\frac{1}{N_k} = \lim_{k\to\infty}\frac{m_k}{N_k} = 0$,

(1.6)

$$w-b \;\leq\; \liminf_{k\to+\infty}\frac{\min\deg_q H_{N_k}(B^{(m_k)})}{m_k(N_k-m_k)} \;\leq\; \limsup_{k\to+\infty}\frac{\min\deg_q H_{N_k}(B^{(m_k)})}{m_k(N_k-m_k)} \;\leq\; w+b.$$

Since the colored $\mathfrak{sl}(N)$ homology categorifies the corresponding colored Reshetikhin-Turaev $\mathfrak{sl}(N)$ polynomial, Theorem 1.2 implies the following colored polynomial Morton-Franks-Williams inequalities.

Corollary 1.3. *Let* B *be a closed braid with writhe* w *and* b *strands. Then*

$$w-b \leq \liminf_{N\to\infty}\frac{\min\deg_q \mathsf{P}_N(B^{(m)})}{m(N-m)} \leq \limsup_{N\to\infty}\frac{\max\deg_q \mathsf{P}_N(B^{(m)})}{m(N-m)} \leq w+b. \tag{1.7}$$

More generally, for any two sequences $\{m_k\}$ *and* $\{N_k\}$ *of positive integers satisfying* $\lim_{k\to\infty}\frac{1}{N_k} = \lim_{k\to\infty}\frac{m_k}{N_k} = 0$,

(1.8)

$$w-b \;\leq\; \liminf_{k\to+\infty}\frac{\min\deg_q \mathsf{P}_{N_k}(B^{(m_k)})}{m_k(N_k-m_k)} \;\leq\; \limsup_{k\to+\infty}\frac{\min\deg_q \mathsf{P}_{N_k}(B^{(m_k)})}{m_k(N_k-m_k)} \;\leq\; w+b.$$

Clearly, (1.5) and (1.7) specialize to (1.3) and (1.2) when $m = 1$. Moreover, Theorem 1.2 and Corollary 1.3 give infinitely many new upper bounds for the self linking number and lower bounds for the braid index.

1.3. **A composition product for the MOY graph polynomial.** To prove the colored Morton-Franks-Williams inequalities, we only need to prove Proposition 1.1. We do so by generalizing the proof of the Morton-Franks-Williams inequality by Jaeger [5]. A key ingredient of our proof is a composition product for the MOY graph polynomial, which generalizes that of Wagner [27]. Before stating our composition product, we briefly recall some basic facts about the MOY graph polynomial [16]. (A more detailed review will be given in Section 2.)

FIGURE 1.

Definition 1.4. An MOY coloring of an oriented trivalent graph is a function from the set of edges of this graph to the set of non-negative integers such that every vertex of the colored graph is of one of the two types in Figure 1.

An MOY graph is an oriented trivalent graph embedded in the plane equipped with an MOY coloring.

For an MOY graph Γ, denote by $E(\Gamma)$ the set of all edges of Γ and by $V(\Gamma)$ the set of all vertices of Γ.

For every positive integer N and every MOY graph Γ, Murakami, Ohtsuki and Yamada [16] defined a single variable polynomial $\langle\Gamma\rangle_N$, which we call the $\mathfrak{sl}(N)$

MOY graph polynomial. To be consistent with the definition of the colored $\mathfrak{sl}(N)$ link homology in [30], we use a slightly different normalization in the present paper. (See Section 2 for more details.)

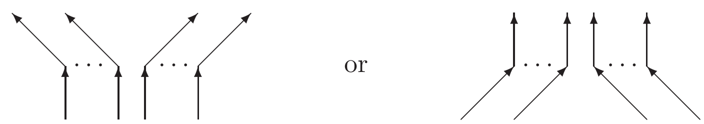


FIGURE 2.

Definition 1.5. Let Γ be an MOY graph. For each edge of Γ with color m, change it into m parallel edges. Also replace each vertex of Γ by one of the two shapes in Figure 2. This changes Γ into a collection $\mathcal{C}$ of oriented circles embedded in the plane. For each circle $C \in \mathcal{C}$, denote by $\mathrm{rot}(C)$ its usual rotation number, that is

$$\mathrm{rot}(C) = \begin{cases} 1 & \text{if } C \text{ is counterclockwise,} \\ -1 & \text{if } C \text{ is clockwise.} \end{cases}$$

Then define

$$\mathrm{rot}(\Gamma) = \sum_{C \in \mathcal{C}} \mathrm{rot}(C). \tag{1.9}$$

Definition 1.6. Let Γ be an MOY graph. Denote by c its color function. That is, for every edge e of Γ, the color of e is $\mathsf{c}(e)$. A labeling f of Γ is an MOY coloring of the underlying oriented trivalent graph of Γ such that $\mathsf{f}(e) \leq \mathsf{c}(e)$ for every edge e of Γ.

Denote by $\mathcal{L}(\Gamma)$ the set of all labellings of Γ. For every $\mathsf{f} \in \mathcal{L}(\Gamma)$, denote by Γ_{f} the MOY graph obtained by re-coloring the underlying oriented trivalent graph of Γ using f.

For every $\mathsf{f} \in \mathcal{L}(\Gamma)$, define a function $\bar{\mathsf{f}}$ on $E(\Gamma)$ by $\bar{\mathsf{f}}(e) = \mathsf{c}(e) - \mathsf{f}(e)$ for every edge e of Γ. It is easy to see that $\bar{\mathsf{f}} \in \mathcal{L}(\Gamma)$.

Let v be a vertex of Γ of either type in Figure 1. (Note that, in either case, e_1 is to the left of e_2 when one looks in the direction of e.) For every $\mathsf{f} \in \mathcal{L}(\Gamma)$, define

$$[v|\Gamma|\mathsf{f}] = \frac{1}{2}(\mathsf{f}(e_1)\bar{\mathsf{f}}(e_2) - \bar{\mathsf{f}}(e_1)\mathsf{f}(e_2)).$$

Theorem 1.7. *Let Γ be an MOY graph. For positive integers M, N and $\mathsf{f} \in \mathcal{L}(\Gamma)$, define*

$$\sigma_{M,N}(\Gamma, \mathsf{f}) = M \cdot \mathrm{rot}(\Gamma_{\bar{\mathsf{f}}}) - N \cdot \mathrm{rot}(\Gamma_{\mathsf{f}}) + \sum_{v \in V(\Gamma)} [v|\Gamma|\mathsf{f}].$$

Then

$$\langle \Gamma \rangle_{M+N} = \sum_{\mathsf{f} \in \mathcal{L}(\Gamma)} q^{\sigma_{M,N}(\Gamma,\mathsf{f})} \cdot \langle \Gamma_{\mathsf{f}} \rangle_M \cdot \langle \Gamma_{\bar{\mathsf{f}}} \rangle_N . \tag{1.10}$$

It is straightforward to check that, if Γ is a $1,2$-colored MOY graph, then (1.10) specializes to Wagner's composition product [27, Lemma 1.1], which implies Jaeger's composition product for the HOMFLY-PT polynomial [5, Proposition 1].

1.4. **Open problems and remarks.** Jaeger's composition product [5] was generalized by Turaev [26] to a comultiplication in the HOMFLY-PT skein module of a thickened surface. Przytycki [20] further proved that this leads to a Hopf algebra structure on this module.

Question 1.8. *Is it possible to interpret the composition product* (1.10) *in a framework similar to that given by Turaev and Przytycki?*

In [30], I constructed a $\mathbb{Z}$-graded $\mathfrak{sl}(N)$ graph homology H_N whose graded dimension is the $\mathfrak{sl}(N)$ MOY graph polynomial. (See [30, Theorem 14.7].) So Theorem 1.7 implies the following corollary, which generalizes [27, Theorem 1.2].

Corollary 1.9. *Let Γ be an MOY graph. Then, for positive integers M and N,*

$$H_{M+N}(\Gamma) \cong \bigoplus_{\mathsf{f}\in\mathcal{L}(\Gamma)} H_M(\Gamma_{\mathsf{f}}) \otimes_{\mathbb{C}} H_N(\Gamma_{\bar{\mathsf{f}}})\{q^{\sigma_{M,N}(\Gamma,\mathsf{f})}\},$$

where the isomorphism preserves the $\mathbb{Z}$-grading.

Question 1.10. *The isomorphism in Corollary 1.9 is obtained here by dimension counting. Is there a more natural construction of this isomorphism? It seems interesting to compare this problem to* [31, Theorem 3.16].

One can modify the definition of the $\mathfrak{sl}(N)$ MOY graph polynomial to get a two-variable MOY graph polynomial and use it to define a colored HOMFLY-PT link polynomial. (See for example [13].) This colored HOMFLY-PT polynomial has been categorified by a colored HOMFLY-PT link homology [28]. We expect Rasmussen's spectral sequence to generalize to a spectral sequence relating the colored HOMFLY-PT link homology to the colored $\mathfrak{sl}(N)$ link homology. This should imply the following conjecture.

Conjecture 1.11. *Denote by H the colored HOMFLY-PT link homology (with an appropriate normalization.) Then*

$$\lim_{N\to\infty} \frac{\min\deg_q H_N(B^{(m)})}{N-m} = \min\deg_x H(B^{(m)}), \tag{1.11}$$

$$\lim_{N\to\infty} \frac{\max\deg_q H_N(B^{(m)})}{N-m} = \max\deg_x H(B^{(m)}), \tag{1.12}$$

where $\deg_x$ is the degree from the x-grading which corresponds to the "framing variable" x of the colored HOMFLY-PT link polynomial.

In particular,

$$w-b \leq \frac{\min\deg_x H(B^{(m)})}{m} \leq \frac{\max\deg_x H(B^{(m)})}{m} \leq w+b. \tag{1.13}$$

Remark 1.12. It seem possible to prove (1.13) without using (1.11) and (1.12). The proof should be a slightly modification of our proof of Theorem 1.2. But this would require a construction of the colored HOMFLY-PT link homology directly modeled on the two-variable MOY graph polynomial.

For any link L, Rutherford [24] proved that $\min\deg_x \mathsf{P}(L) = w-b$ for some braid representation of L with writhe w and b strands if and only if there exists a Legendrian front projection of L that has an oriented ruling.

Question 1.13. *Can one generalize Rutherford's result to a necessary and sufficient condition to the sharpness of any of* (1.5), (1.6), (1.7) *or* (1.8)*?*

Applying the Morton-Franks-Williams inequality to cables, Morton and Short [15] introduced the cabled Morton-Franks-Williams inequalities. For simplicity, let us only consider knots. Suppose a knot K has a braid diagram of b strands with writhe w. Denote by $K_{m,k}$ the (m,k)-cable of K. Then the braid diagram of K leads to an obvious braid diagram of $K_{m,k}$ of mb strands with writhe $mw+k(m-1)$. Applying (1.3) to this braid diagram of $K_{m,k}$, one gets

$$w-b \leq \liminf_{N\to\infty} \frac{\min\deg_q H_N(K_{m,k})}{m(N-1)} - \frac{k(m-1)}{m} \leq \limsup_{N\to\infty} \frac{\max\deg_q H_N(K_{m,k})}{m(N-1)} - \frac{k(m-1)}{m} \leq w+b. \tag{1.14}$$

Question 1.14. *How does the colored inequality* (1.5) *compare to the cabled inequality* (1.14)*? Is there an explicit relation between the $\mathfrak{sl}(N)$ homology of $K^{(m)}$ and that of $K_{m,k}$? Will we get better bounds by applying the colored Morton-Franks-Williams inequalities to cables?*

One interesting application of the Morton-Franks-Williams inequality is to verify the following Jones Conjecture for special class of links.

Conjecture 1.15. [6, end of Section 8] *For any link L of braid index b, if B_1 and B_2 are two braid diagrams of L of b strands, then the writhes of B_1 and B_2 are equal.*

If both ends of the Morton-Franks-Williams inequality (1.1) are sharp for a closed braid B, then half of the x-span of the HOMFLY-PT polynomial is equal to the braid index of B and the Jones Conjecture is true for B. But, since the Morton-Franks-Williams inequality is in general not sharp, this argument works only for special classes of links. See for example [3, 9, 12, 17, 18, 25] for related results.

Question 1.16. *It is clear that the sharpness (of both ends) of any of the inequalities* (1.5)*,* (1.6)*,* (1.7) *or* (1.8) *would similarly imply the Jones Conjecture. Can one use these inequalities to obtain further results on the Jones Conjecture?*

1.5. **Organization of this paper.** The construction of the colored $\mathfrak{sl}(N)$ link homology is used only superficially in the present paper. So no prior experience in the colored $\mathfrak{sl}(N)$ link homology is needed to understand the proofs in this paper. In Section 2, we will review aspects of the colored $\mathfrak{sl}(N)$ link polynomial and homology that are used in our proofs. We will then establish the composition product in Section 3 and apply it to prove the colored Morton-Franks-Williams inequalities in Section 4.

Acknowledgments. I would like to thank Jozef Przytycki for interesting discussions on the history of the composition product.

2. The Colored $\mathfrak{sl}(N)$ Link Polynomial and Homology

Using the MOY graph polynomial, Murakami, Ohtsuki and Yamada [16] gave an alternative construction of the $\mathfrak{sl}(N)$ Reshetikhin-Turaev polynomial [23] for links colored by non-negative integers. We now briefly review their construction in [16] and the definition of the colored $\mathfrak{sl}(N)$ link homology in [30].

2.1. **The MOY graph polynomial.** We review the MOY graph polynomial [16] in this subsection. Our notations and normalizations are slightly different from that used in [16].

For a positive integer N, define $\Sigma_N = \{2k - N + 1 | k = 0, 1, \ldots, N - 1\}$. Denote by $\mathcal{P}(\Sigma_N)$ the set of subsets of Σ_N. For a finite set A, denote by $\#A$ the cardinality of A. Define a function $\pi : \mathcal{P}(\Sigma_N) \times \mathcal{P}(\Sigma_N) \to \mathbb{Z}_{\geq 0}$ by

$$\pi(A_1, A_2) = \#\{(a_1, a_2) \in A_1 \times A_2 \mid a_1 > a_2\} \text{ for } A_1,\ A_2 \in \mathcal{P}(\Sigma_N). \tag{2.1}$$

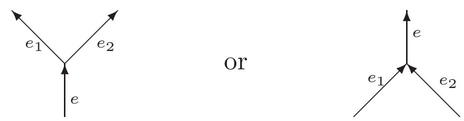


Figure 3.

Let Γ be an MOY graph. Denote by $E(\Gamma)$ the set of edges of Γ, by $V(\Gamma)$ the set of vertices of Γ and by $\mathsf{c} : E(\Gamma) \to \mathbb{Z}_{\geq 0}$ the color function of Γ. That is, for every edge e of Γ, $\mathsf{c}(e) \in \mathbb{Z}_{\geq 0}$ is the color of e.

A state of Γ is a function $\varphi : E(\Gamma) \to \mathcal{P}(\Sigma_N)$ such that

(i) for every edge e of Γ, $\#\varphi(e) = \mathsf{c}(e)$,
(ii) for every vertex v of Γ, as depicted in Figure 3, we have $\varphi(e) = \varphi(e_1) \cup \varphi(e_2)$.

Note that (i) and (ii) imply that $\varphi(e_1) \cap \varphi(e_2) = \emptyset$.

Denote by $\mathcal{S}_N(\Gamma)$ the set of states of Γ.

For a state φ of Γ and a vertex v of Γ (as depicted in Figure 3), the weight of v with respect to φ is defined to be

$$\mathrm{wt}(v; \varphi) = q^{\frac{\mathsf{c}(e_1)\mathsf{c}(e_2)}{2} - \pi(\varphi(e_1), \varphi(e_2))}. \tag{2.2}$$

Given a state φ of Γ, replace each edge e of Γ by $\mathsf{c}(e)$ parallel edges, assign to each of these new edges a different element of $\varphi(e)$ and, at every vertex, connect each pair of new edges assigned the same element of Σ_N. This changes Γ into a collection $\mathcal{C}_\varphi$ of embedded circles, each of which is assigned an element of Σ_N. By abusing notation, we denote by $\varphi(C)$ the element of Σ_N assigned to $C \in \mathcal{C}_\varphi$. Note that:

- There may be intersections between different circles in $\mathcal{C}_\varphi$. But, each circle in $\mathcal{C}_\varphi$ is embedded, that is, it has no self-intersections or self-tangencies.
- There may be more than one way to do this. But if we view $\mathcal{C}_\varphi$ as a virtue link and the intersection points between different elements of $\mathcal{C}_\varphi$ virtual crossings, then the above construction is unique up to purely virtual regular Reidemeister moves.

The rotation number $\mathrm{rot}(\varphi)$ of φ is then defined to be

$$\mathrm{rot}(\varphi) = \sum_{C \in \mathcal{C}_\varphi} \varphi(C)\mathrm{rot}(C). \tag{2.3}$$

Clearly, $\mathrm{rot}(\varphi)$ is independent of the choices made in its definition. We also make the following simple observation, which will be useful in Section 3.

Lemma 2.1. *For any state φ of Γ,*

$$\sum_{C\in\mathcal{C}_\varphi} \mathrm{rot}(C) = \mathrm{rot}(\Gamma).$$

Now we are ready to define the $\mathfrak{sl}(N)$ MOY graph polynomial.

Definition 2.2. [16] The $\mathfrak{sl}(N)$ MOY graph polynomial of Γ is defined to be

$$\langle\Gamma\rangle_N := \sum_{\varphi\in\mathcal{S}_N(\Gamma)} \Big(\prod_{v\in V(\Gamma)} \mathrm{wt}(v;\varphi)\Big) q^{\mathrm{rot}(\varphi)}. \tag{2.4}$$

Remark 2.3. 1. If Γ contains an edge with color greater than N, then $\mathcal{S}_N(\Gamma)=\emptyset$ and therefore, $\langle\Gamma\rangle_N = 0$.
2. If Γ contains an edge with color 0, then erasing this edge does not change the polynomial $\langle\Gamma\rangle_N$.
3. We allow Γ to be the empty graph and use the convention $\langle\emptyset\rangle_N = 1$.

2.2. **The colored $\mathfrak{sl}(N)$ link polynomial.** The $\mathfrak{sl}(N)$ Reshetikhin-Turaev polynomial [23] for links colored by non-negative integers can be expressed as a combination of the $\mathfrak{sl}(N)$ MOY graph polynomials of the MOY resolutions of its diagram.

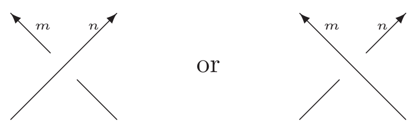


FIGURE 4.

Let D be a diagram of a link colored by non-negative integers. An MOY resolution of D is an MOY graph obtained by replacing each crossing of D (as shown in Figure 4) by the shape in Figure 5 for some integer k satisfying $\max\{0, m-n\} \leq k \leq m$. Denote by $\mathcal{R}(D)$ the set of all MOY resolutions of D.

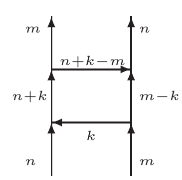
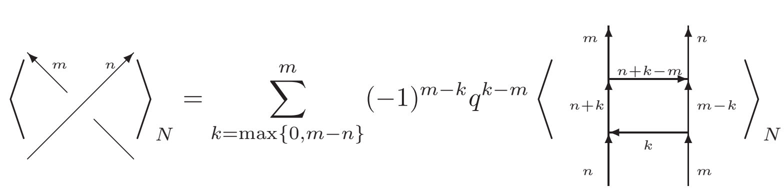
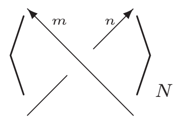


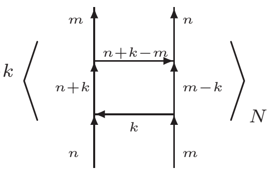
FIGURE 5.

Definition 2.4. [16] For a link diagram D colored by non-negative integers, define the unnormalized Reshetikhin-Turaev $\mathfrak{sl}(N)$ polynomial $\langle D\rangle_N$ of D by applying the following skein sum at every crossing of D.

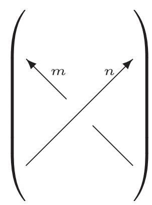




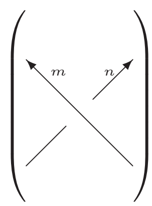
$$= \sum_{k=\max\{0,m-n\}}^{m} (-1)^{k-m} q^{m-k}$$



Also, for each crossing c of D, define the shifting factor $\mathsf{s}(c)$ of c by

s 

$$= \begin{cases} (-1)^{-m} q^{m(N+1-m)} & \text{if } m = n, \\ 1 & \text{if } m \neq n, \end{cases}$$

s 

$$= \begin{cases} (-1)^{m} q^{-m(N+1-m)} & \text{if } m = n, \\ 1 & \text{if } m \neq n. \end{cases}$$

The normalized Reshetikhin-Turaev $\mathfrak{sl}(N)$ polynomial $\mathsf{P}_N(D)$ of D is defined to be

$$\mathsf{P}_N(D) = \langle D \rangle_N \cdot \prod_c \mathsf{s}(c),$$

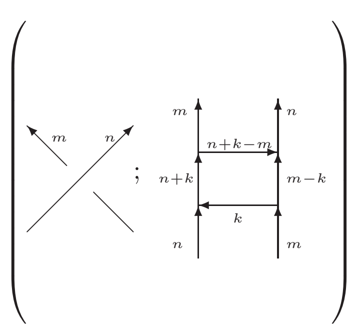
where c runs through all crossings of D.

Theorem 2.5. [16] *$\langle D \rangle_N$ is invariant under Reidemeister moves (II) and (III). $\mathsf{P}_N(D)$ is invariant under all Reidemeister moves.*

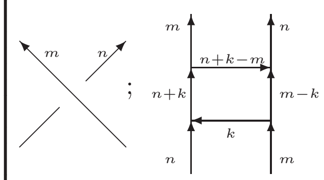
2.3. **The colored $\mathfrak{sl}(N)$ link homology.** For a positive integer N and an MOY graph Γ, I defined in [30] a $\mathbb{Z}$-graded homology $H_N(\Gamma)$ whose graded dimension is $\langle \Gamma \rangle_N$. We call this grading the quantum grading of $H_N(\Gamma)$.

Next we give $H_N(\Gamma)$ a homological grading such that the whole of $H_N(\Gamma)$ has homological grading 0. This makes it a $\mathbb{Z}^{\oplus 2}$-graded space.

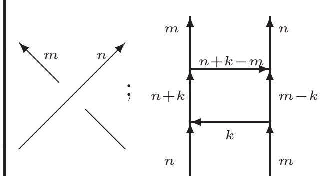
For the resolution of a crossing, define

(2.5) $\mathsf{s}_{h,N}$ 

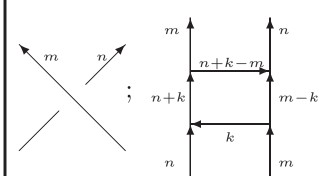
$$= \begin{cases} -k & \text{if } m = n, \\ m-k & \text{if } m \neq n, \end{cases}$$

(2.6) $\mathsf{s}_{h,N}$ 

$$= \begin{cases} k & \text{if } m = n, \\ k - m & \text{if } m \neq n, \end{cases}$$

(2.7) $\mathsf{s}_{q,N}$ 

$$= \begin{cases} k - m + m(N+1-m) & \text{if } m = n, \\ k - m & \text{if } m \neq n, \end{cases}$$

(2.8) $\mathsf{s}_{q,N}$ 

$$= \begin{cases} m - k - m(N+1-m) & \text{if } m = n, \\ m - k & \text{if } m \neq n. \end{cases}$$

For a link diagram D colored by non-negative integers and an MOY resolution Γ of D, define $\mathsf{s}_{h,N}(D;\Gamma)$ to be the sum of the values of $\mathsf{s}_{h,N}$ over all crossings of D and define $\mathsf{s}_{q,N}(D;\Gamma)$ to be the sum of the values of $\mathsf{s}_{q,N}$ over all crossings of D.

Denote by $H_N(\Gamma)\|\mathsf{s}_{h,N}(D;\Gamma)\|\{q^{\mathsf{s}_{q,N}(D;\Gamma)}\}$ the space obtained from $H_N(\Gamma)$ by shifting its homological grading by $\mathsf{s}_{h,N}(D;\Gamma)$ and shifting its quantum grading by $\mathsf{s}_{q,N}(D;\Gamma)$.

Theorem 2.6. [30] *Let D be a link diagram colored by non-negative integers. Then one can equip the $\mathbb{Z}^{\oplus 2}$-graded space*

$$\bigoplus_{\Gamma \in \mathcal{R}(D)} H_N(\Gamma)\|\mathsf{s}_{h,N}(D;\Gamma)\|\{q^{\mathsf{s}_{q,N}(D;\Gamma)}\}$$

with a homogeneous differential map of quantum grading 0 and homological grading 1 so that the homology of this chain complex, with its $\mathbb{Z}^{\oplus 2}$-grading, is invariant under all Reidemeister moves.

We denote this invariant link homology by H_N. From the form of the above chain complex, one can see that the graded Euler characteristic of $H_N(D)$ is the normalized Reshetikhin-Turaev $\mathfrak{sl}(N)$ polynomial $\mathsf{P}_N(D)$.

3. Proof of The Composition Product

Jaeger [5] proved his composition product formula by showing that the composition product satisfies the skein relation that uniquely characterizes the HOMFLY-PT polynomial. Similarly, Wagner [27] proved his composition product formula by showing that the composition product satisfies the MOY relations that uniquely characterizes the $1,2$-colored MOY graph polynomial. The proof of Theorem 1.7 would be rather lengthy if we use a direct generalization of their approach. Fortunately, the composition product (1.10) in Theorem 1.7 is a simple corollary of the MOY state sum formula (2.4), which makes the proof a lot easier. In fact, it is not hard to see that (1.10) and (2.4) are actually equivalent to each other.

Definition 3.1. Let Γ be an MOY graph, and M, N two positive integers. For a state $\varphi \in \mathcal{S}_{M+N}(\Gamma)$, define $\varphi_1 : E(\Gamma) \to \mathcal{P}(\Sigma_M)$ by

$$\varphi_1(e) = \{k \mid k - N \in \varphi(e) \cap \{2k - M - N + 1 \mid k = 0, 1, \dots, M - 1\}\}$$

and $\varphi_2 : E(\Gamma) \to \mathcal{P}(\Sigma_N)$ by

$$\varphi_2(e) = \{k \mid k + M \in \varphi(e) \cap \{2k - M - N + 1 \mid k = M, M + 1, \dots, M + N - 1\}\}.$$

Moreover, define $\mathsf{f}_\varphi : E(\Gamma) \to \mathbb{Z}_{\geq 0}$ by $\mathsf{f}_\varphi(e) = \#\varphi_1(e)$.

It is easy to see that $\mathsf{f}_\varphi \in \mathcal{L}(\Gamma)$, $\varphi_1 \in \mathcal{S}_M(\Gamma_{\mathsf{f}_\varphi})$ and $\varphi_2 \in \mathcal{S}_N(\Gamma_{\bar{\mathsf{f}}_\varphi})$.

For $\mathsf{f} \in \mathcal{L}(\Gamma)$, define $\mathcal{S}^{\mathsf{f}}_{M+N}(\Gamma) = \{\varphi \in \mathcal{S}_{M+N}(\Gamma) \mid \mathsf{f}_\varphi = \mathsf{f}\}$.

Lemma 3.2.

$$\mathcal{S}_{M+N}(\Gamma) = \bigsqcup_{\mathsf{f} \in \mathcal{L}(\Gamma)} \mathcal{S}^{\mathsf{f}}_{M+N}(\Gamma)$$

and therefore

$$\langle \Gamma \rangle_{M+N} = \sum_{\mathsf{f} \in \mathcal{L}(\Gamma)} \sum_{\varphi \in \mathcal{S}^{\mathsf{f}}_{M+N}(\Gamma)} \Big(\prod_{v \in V(\Gamma)} \mathrm{wt}(v;\varphi)\Big) q^{\mathrm{rot}(\varphi)}. \tag{3.1}$$

Proof. This lemma follows easily from the relevant definitions. We leave the details to the reader. □

Lemma 3.3. *For any $\mathsf{f} \in \mathcal{L}(\Gamma)$, the function $\mathcal{S}^{\mathsf{f}}_{M+N}(\Gamma) \to \mathcal{S}_M(\Gamma_{\mathsf{f}}) \times \mathcal{S}_N(\Gamma_{\bar{\mathsf{f}}})$ given by $\varphi \mapsto (\varphi_1, \varphi_2)$ is a bijection. Moreover, for any $\varphi \in \mathcal{S}^{\mathsf{f}}_{M+N}(\Gamma)$,*

$$\mathrm{rot}(\varphi) = \mathrm{rot}(\varphi_1) + \mathrm{rot}(\varphi_2) - N \cdot \mathrm{rot}(\Gamma_{\mathsf{f}}) + M \cdot \mathrm{rot}(\Gamma_{\bar{\mathsf{f}}}), \tag{3.2}$$

$$\mathrm{wt}(v;\varphi) = \mathrm{wt}(v;\varphi_1) \cdot \mathrm{wt}(v;\varphi_2) \cdot q^{[v|\Gamma|\mathsf{f}]}, \tag{3.3}$$

where v is any vertex of Γ.

Proof. By the definition of φ_1 and φ_2, it is clear that $\varphi \mapsto (\varphi_1, \varphi_2)$ gives a bijection $\mathcal{S}^{\mathsf{f}}_{M+N}(\Gamma) \to \mathcal{S}_M(\Gamma_{\mathsf{f}}) \times \mathcal{S}_N(\Gamma_{\bar{\mathsf{f}}})$. Equation (3.2) follows easily from Lemma 2.1 and the definitions of rotation numbers (equations (1.9) and (2.3).) It remains to prove (3.3).

Denote by c the color function of Γ. Then $\mathsf{c} = \mathsf{f} + \bar{\mathsf{f}}$. Let v be a vertex of Γ as shown in Figure 1. Recall that

$$\begin{aligned}
\mathrm{wt}(v;\varphi) &= q^{\frac{\mathsf{c}(e_1)\mathsf{c}(e_2)}{2} - \pi(\varphi(e_1),\varphi(e_2))}, \\
\mathrm{wt}(v;\varphi_1) &= q^{\frac{\mathsf{f}(e_1)\mathsf{f}(e_2)}{2} - \pi(\varphi_1(e_1),\varphi_1(e_2))}, \\
\mathrm{wt}(v;\varphi_2) &= q^{\frac{\bar{\mathsf{f}}(e_1)\bar{\mathsf{f}}(e_2)}{2} - \pi(\varphi_2(e_1),\varphi_2(e_2))},
\end{aligned}$$

where π is defined in (2.1). Let

$$\begin{aligned}\Sigma' &= \{2k-M-N+1 \mid k=0,1,\ldots,M-1\},\\ \Sigma'' &= \{2k-M-N+1 \mid k=M,M+1,\ldots,M+N-1\}.\end{aligned}$$

Then $\Sigma_{M+N} = \Sigma' \sqcup \Sigma''$ and

$$\begin{aligned}& \pi(\varphi(e_1),\varphi(e_2))\\ = {} & \pi((\varphi(e_1)\cap\Sigma')\sqcup(\varphi(e_1)\cap\Sigma''),(\varphi(e_2)\cap\Sigma')\sqcup(\varphi(e_2)\cap\Sigma''))\\ = {} & \pi(\varphi(e_1)\cap\Sigma',\varphi(e_2)\cap\Sigma')+\pi(\varphi(e_1)\cap\Sigma'',\varphi(e_2)\cap\Sigma'')\\ & +\pi(\varphi(e_1)\cap\Sigma'',\varphi(e_2)\cap\Sigma')+\pi(\varphi(e_1)\cap\Sigma',\varphi(e_2)\cap\Sigma'')\\ = {} & \pi(\varphi_1(e_1),\varphi_1(e_2))+\pi(\varphi_2(e_1),\varphi_2(e_2))+\bar{\mathsf{f}}(e_1)\mathsf{f}(e_2).\end{aligned}$$

Thus,

$$\begin{aligned}& \left(\frac{\mathsf{c}(e_1)\mathsf{c}(e_2)}{2}-\pi(\varphi(e_1),\varphi(e_2))\right)-\left(\frac{\mathsf{f}(e_1)\mathsf{f}(e_2)}{2}-\pi(\varphi_1(e_1),\varphi_1(e_2))\right)\\ & -\left(\frac{\bar{\mathsf{f}}(e_1)\bar{\mathsf{f}}(e_2)}{2}-\pi(\varphi_2(e_1),\varphi_2(e_2))\right)\\ = {} & \frac{(\mathsf{f}(e_1)+\bar{\mathsf{f}}(e_1))(\mathsf{f}(e_2)+\bar{\mathsf{f}}(e_2))-\mathsf{f}(e_1)\mathsf{f}(e_2)-\bar{\mathsf{f}}(e_1)\bar{\mathsf{f}}(e_2)}{2}-\bar{\mathsf{f}}(e_1)\mathsf{f}(e_2)\\ = {} & \frac{1}{2}(\mathsf{f}(e_1)\bar{\mathsf{f}}(e_2)-\bar{\mathsf{f}}(e_1)\mathsf{f}(e_2)) = [v|\Gamma|\mathsf{f}].\end{aligned}$$

This proves (3.3). □

Theorem 1.7 follows easily from Lemmas 3.2 and 3.3.

Proof of Theorem 1.7. By Lemmas 3.2 and 3.3,

$$\begin{aligned}& \langle\Gamma\rangle_{M+N}\\ = {} & \sum_{\mathsf{f}\in\mathcal{L}(\Gamma)}\sum_{\varphi\in\mathcal{S}^{\mathsf{f}}_{M+N}(\Gamma)}\left(\prod_{v\in V(\Gamma)}\mathrm{wt}(v;\varphi)\right)q^{\mathrm{rot}(\varphi)}\\ = {} & \sum_{\mathsf{f}\in\mathcal{L}(\Gamma)}\sum_{\varphi\in\mathcal{S}^{\mathsf{f}}_{M+N}(\Gamma)}\left(\prod_{v\in V(\Gamma)}\mathrm{wt}(v;\varphi_1)\cdot\mathrm{wt}(v;\varphi_2)\cdot q^{[v|\Gamma|\mathsf{f}]}\right)q^{\mathrm{rot}(\varphi_1)+\mathrm{rot}(\varphi_2)-N\cdot\mathrm{rot}(\Gamma_{\mathsf{f}})+M\cdot\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})}\\ = {} & \sum_{\mathsf{f}\in\mathcal{L}(\Gamma)}q^{\sigma_{M,N}(\Gamma,\mathsf{f})}\sum_{(\varphi_1,\varphi_2)\in\mathcal{S}_M(\Gamma_{\mathsf{f}})\times\mathcal{S}_N(\Gamma_{\bar{\mathsf{f}}})}\left(\prod_{v\in V(\Gamma)}\mathrm{wt}(v;\varphi_1)\cdot q^{\mathrm{rot}(\varphi_1)}\right)\cdot\left(\prod_{v\in V(\Gamma)}\mathrm{wt}(v;\varphi_2)\cdot q^{\mathrm{rot}(\varphi_2)}\right)\\ = {} & \sum_{\mathsf{f}\in\mathcal{L}(\Gamma)}q^{\sigma_{M,N}(\Gamma,\mathsf{f})}\cdot\langle\Gamma_{\mathsf{f}}\rangle_M\cdot\langle\Gamma_{\bar{\mathsf{f}}}\rangle_N\,.\end{aligned}$$

□

Remark 3.4. The above proof shows that the MOY state sum formula (2.4) implies the composition product (1.10). Now consider $\langle\Gamma\rangle_N$ as $\langle\Gamma\rangle_{\underbrace{1+\cdots+1}_{N\ 1's}}$ and use the composition product (1.10) repeatedly. This give a state sum formula of $\langle\Gamma\rangle_N$ in terms of $\langle\rangle_1$ of a family of simple MOY graphs, each of which is a collection of embedded circles colored by 1. (A special case of the formula is given in [27].) It is not very hard to see that this state sum formula is exactly the MOY state sum formula (2.4). Thus (1.10) and (2.4) are equivalent.

4. Proof of the Colored Morton-Franks-Williams Inequalities

We prove in this section Proposition 1.1, which implies the colored Morton-Franks-Williams inequalities (1.5), (1.6), (1.7) or (1.8). We do so by establish upper and lower bounds for the degree of the $\mathfrak{sl}(N)$ MOY graph polynomial of a special type of MOY graphs, which we call MOY tracks.

4.1. **MOY tracks.**

Definition 4.1. Let b be a positive integer. Assume $0 \leq x_1 < \cdots < x_b \leq 1$. An MOY track of b strands is an MOY graph Γ satisfying:

1. The part of Γ outside $[0,1] \times [0,1]$ consists of b edges such that, for each $i = 1, \ldots, b$, one of these b edges connects $(x_i, 0)$ to $(x_i, 1)$ via the right side of $[0,1] \times [0,1]$.
2. The part of Γ inside $[0,1] \times [0,1]$ consists of vertical and horizontal edges only.
3. All vertical edges of Γ inside $[0,1] \times [0,1]$ point downward.
4. The union of all vertical edges of Γ inside $[0,1] \times [0,1]$, as a point set, is the set $\{x_1, \ldots, x_b\} \times [0,1]$.
5. Each horizontal edge inside $[0,1] \times [0,1]$ starts and ends on adjacent vertical edges inside $[0,1] \times [0,1]$.

Note that the MOY resolutions of a braid diagram colored by non-negative integers are all MOY tracks.

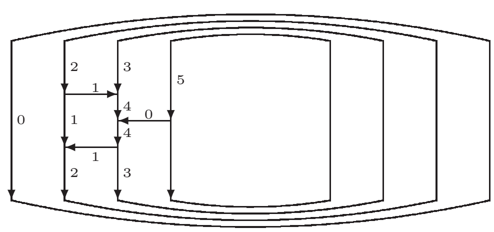


Figure 6. An MOY track of 4 strands

The goal of this subsection is to prove Proposition 4.3, which gives us upper and lower bounds for the degree of the $\mathfrak{sl}(N)$ MOY graph polynomial of an MOY track. For simplicity, we introduce the following notations.

Definition 4.2. Let Γ be an MOY graph and v a vertex of Γ as shown in Figure 1. Define $\rho_\Gamma(v) = \frac{mn}{2}$ and

$$\rho(\Gamma) = \sum_{v \in V(\Gamma)} \rho_\Gamma(v).$$

Proposition 4.3. *Let Γ be an MOY track of b strands colored by $0, 1, \ldots, N$. Then*

(4.1)
$$-\mathrm{rot}(\Gamma)(N - \frac{\mathrm{rot}(\Gamma)}{b}) - \rho(\Gamma) \leq \min \deg_q \langle \Gamma \rangle_N \leq \max \deg_q \langle \Gamma \rangle_N \leq \mathrm{rot}(\Gamma)(N - \frac{\mathrm{rot}(\Gamma)}{b}) + \rho(\Gamma).$$

Proof. Note that $\rho(\Gamma) \geq 0$. Also, since $\frac{\mathrm{rot}(\Gamma)}{b}$ is the average of the colors of of the b edges of Γ outside $[0,1] \times [0,1]$, we have $0 \leq \frac{\mathrm{rot}(\Gamma)}{b} \leq N$.

We prove inequality (4.1) by an induction on N. If $N = 1$, then Γ is colored by $0, 1$. It is then easy to check that $\rho(\Gamma) = 0$, $\langle \Gamma \rangle_N = 1$ and therefore, $\min \deg_q \langle \Gamma \rangle_N = \max \deg_q \langle \Gamma \rangle_N = 0$. This implies that (4.1) holds for $N = 1$.

Now assume (4.1) is true for N. Suppose Γ is an MOY track colored by $0, 1, \dots, N+1$. Using the composition product (Theorem 1.7,) we get

$$\langle \Gamma \rangle_{N+1} = \sum_{\mathsf{f} \in \mathcal{L}(\Gamma)} q^{\sigma_{N,1}(\Gamma,\mathsf{f})} \cdot \langle \Gamma_{\mathsf{f}} \rangle_N \cdot \langle \Gamma_{\bar{\mathsf{f}}} \rangle_1 . \tag{4.2}$$

Clearly, for a term $q^{\sigma_{N,1}(\Gamma,\mathsf{f})} \cdot \langle \Gamma_{\mathsf{f}} \rangle_N \cdot \langle \Gamma_{\bar{\mathsf{f}}} \rangle_1$ in (4.2) to be non-zero, f must satisfy that $0 \leq \mathsf{f}(e) \leq N$ and $0 \leq \bar{\mathsf{f}}(e) \leq 1$ for all $e \in E(\Gamma)$, which implies that $0 \leq \frac{\mathrm{rot}(\Gamma_{\mathsf{f}})}{b} \leq N$ and $0 \leq \frac{\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})}{b} \leq 1$. Moreover, by the induction hypothesis, (4.1) is true for $\langle \Gamma_{\mathsf{f}} \rangle_N$ and $\langle \Gamma_{\bar{\mathsf{f}}} \rangle_1$.

By direct computation, we get

$$\begin{aligned}
& -\mathrm{rot}(\Gamma)(N+1-\frac{\mathrm{rot}(\Gamma)}{b}) + \mathrm{rot}(\Gamma_{\mathsf{f}})(N-\frac{\mathrm{rot}(\Gamma_{\mathsf{f}})}{b}) + \mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})(1-\frac{\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})}{b}) \\
= \;& -2 \cdot \mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})(N - \mathrm{rot}(\Gamma_{\mathsf{f}})) + N\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}}) - \mathrm{rot}(\Gamma_{\mathsf{f}}) \\
\leq \;& N\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}}) - \mathrm{rot}(\Gamma_{\mathsf{f}}).
\end{aligned}$$

So

$$-\mathrm{rot}(\Gamma)(N+1-\frac{\mathrm{rot}(\Gamma)}{b}) \leq -\mathrm{rot}(\Gamma_{\mathsf{f}})(N-\frac{\mathrm{rot}(\Gamma_{\mathsf{f}})}{b}) - \mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})(1-\frac{\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})}{b}) + N\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}}) - \mathrm{rot}(\Gamma_{\mathsf{f}}). \tag{4.3}$$

Similarly,

$$\begin{aligned}
& \mathrm{rot}(\Gamma)(N+1-\frac{\mathrm{rot}(\Gamma)}{b}) - \mathrm{rot}(\Gamma_{\mathsf{f}})(N-\frac{\mathrm{rot}(\Gamma_{\mathsf{f}})}{b}) - \mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})(1-\frac{\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})}{b}) \\
= \;& 2 \cdot \mathrm{rot}(\Gamma_{\mathsf{f}})(1-\frac{\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})}{b}) + N\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}}) - \mathrm{rot}(\Gamma_{\mathsf{f}}) \\
\geq \;& N\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}}) - \mathrm{rot}(\Gamma_{\mathsf{f}}).
\end{aligned}$$

So

$$\mathrm{rot}(\Gamma)(N+1-\frac{\mathrm{rot}(\Gamma)}{b}) \geq \mathrm{rot}(\Gamma_{\mathsf{f}})(N-\frac{\mathrm{rot}(\Gamma_{\mathsf{f}})}{b}) + \mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})(1-\frac{\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})}{b}) + N\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}}) - \mathrm{rot}(\Gamma_{\mathsf{f}}). \tag{4.4}$$

For every $v \in V(\Gamma)$ as depicted in Figure 1, we have

$$\rho_\Gamma(v) - \rho_{\Gamma_{\mathsf{f}}}(v) - \rho_{\Gamma_{\bar{\mathsf{f}}}}(v) = \frac{1}{2}(\mathsf{f}(e_1)\bar{\mathsf{f}}(e_2) + \bar{\mathsf{f}}(e_1)\mathsf{f}(e_2)) \geq 0.$$

Recall that

$$[v|\Gamma|\mathsf{f}] = \frac{1}{2}(\mathsf{f}(e_1)\bar{\mathsf{f}}(e_2) - \bar{\mathsf{f}}(e_1)\mathsf{f}(e_2)).$$

This implies that

$$-(\rho_\Gamma(v) - \rho_{\Gamma_{\mathsf{f}}}(v) - \rho_{\Gamma_{\bar{\mathsf{f}}}}(v)) \leq [v|\Gamma|\mathsf{f}] \leq \rho_\Gamma(v) - \rho_{\Gamma_{\mathsf{f}}}(v) - \rho_{\Gamma_{\bar{\mathsf{f}}}}(v). \tag{4.5}$$

Putting (4.3), (4.4) and (4.5) together, we get that

$$\begin{aligned} & -\mathrm{rot}(\Gamma)(N+1-\frac{\mathrm{rot}(\Gamma)}{b})-\rho(\Gamma) \\ \leq\ & \sigma_{N,1}(\Gamma,\mathsf{f})-\mathrm{rot}(\Gamma_{\mathsf{f}})(N-\frac{\mathrm{rot}(\Gamma_{\mathsf{f}})}{b})-\rho(\Gamma_{\mathsf{f}})-\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})(1-\frac{\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})}{b})-\rho(\Gamma_{\bar{\mathsf{f}}}) \end{aligned}$$

and

$$\begin{aligned} & \mathrm{rot}(\Gamma)(N+1-\frac{\mathrm{rot}(\Gamma)}{b})+\rho(\Gamma) \\ \geq\ & \sigma_{N,1}(\Gamma,\mathsf{f})+\mathrm{rot}(\Gamma_{\mathsf{f}})(N-\frac{\mathrm{rot}(\Gamma_{\mathsf{f}})}{b})+\rho(\Gamma_{\mathsf{f}})+\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})(1-\frac{\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})}{b})+\rho(\Gamma_{\bar{\mathsf{f}}}). \end{aligned}$$

But (4.1) is true for $\langle\Gamma_{\mathsf{f}}\rangle_N$ and $\langle\Gamma_{\bar{\mathsf{f}}}\rangle_1$. That is,

$$\begin{aligned} \min\deg_q\langle\Gamma_{\mathsf{f}}\rangle_N &\geq -\mathrm{rot}(\Gamma_{\mathsf{f}})(N-\frac{\mathrm{rot}(\Gamma_{\mathsf{f}})}{b})-\rho(\Gamma_{\mathsf{f}}), \\ \max\deg_q\langle\Gamma_{\mathsf{f}}\rangle_N &\leq \mathrm{rot}(\Gamma_{\mathsf{f}})(N-\frac{\mathrm{rot}(\Gamma_{\mathsf{f}})}{b})+\rho(\Gamma_{\mathsf{f}}), \\ \min\deg_q\langle\Gamma_{\bar{\mathsf{f}}}\rangle_1 &\geq -\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})(1-\frac{\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})}{b})-\rho(\Gamma_{\bar{\mathsf{f}}}), \\ \max\deg_q\langle\Gamma_{\bar{\mathsf{f}}}\rangle_1 &\leq \mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})(1-\frac{\mathrm{rot}(\Gamma_{\bar{\mathsf{f}}})}{b})+\rho(\Gamma_{\bar{\mathsf{f}}}). \end{aligned}$$

Thus, for any non-zero term $q^{\sigma_{N,1}(\Gamma,\mathsf{f})}\cdot\langle\Gamma_{\mathsf{f}}\rangle_N\cdot\langle\Gamma_{\bar{\mathsf{f}}}\rangle_1$ in (4.2), we have

$$\begin{aligned} \min\deg_q(q^{\sigma_{N,1}(\Gamma,\mathsf{f})}\cdot\langle\Gamma_{\mathsf{f}}\rangle_N\cdot\langle\Gamma_{\bar{\mathsf{f}}}\rangle_1) &\geq -\mathrm{rot}(\Gamma)(N+1-\frac{\mathrm{rot}(\Gamma)}{b})-\rho(\Gamma), \\ \max\deg_q(q^{\sigma_{N,1}(\Gamma,\mathsf{f})}\cdot\langle\Gamma_{\mathsf{f}}\rangle_N\cdot\langle\Gamma_{\bar{\mathsf{f}}}\rangle_1) &\leq \mathrm{rot}(\Gamma)(N+1-\frac{\mathrm{rot}(\Gamma)}{b})+\rho(\Gamma). \end{aligned}$$

This shows that (4.1) is true for $\langle\Gamma\rangle_{N+1}$ and completes the induction. □

4.2. **Proof of Proposition 1.1.** Proposition 1.1 is now a simple corollary of Proposition 4.3. All we need to do is to keep track of the grading shifts used in the definition of the colored $\mathfrak{sl}(N)$ link homology.

Proof of Proposition 1.1. Denote by $c_1,\dots,c_l$ the l crossings of $B^{(m)}$ and by $\varepsilon_i=\pm1$ the sign of the crossing c_i for $i=1,\dots,l$. We say that replacing a crossing in Figure 4 by the shape in Figure 5 is a k-resolution of the crossing. Denote by $\Gamma_{k_1,\dots,k_l}$ the MOY resolution of $B^{(m)}$ obtained by applying the k_i resolution on c_i for $i=1,\dots,l$. Note that $\mathrm{rot}(\Gamma_{k_1,\dots,k_l})=bm$. Moreover, each crossing c_i in $B^{(m)}$ gives rise to four vertices in $\Gamma_{k_1,\dots,k_l}$. The $\rho_{\Gamma_{k_1,\dots,k_l}}$ value of these four vertices are $\frac{k_im}{2}$, $\frac{k_im}{2}$, $\frac{k_i(m-k_i)}{2}$ and $\frac{k_i(m-k_i)}{2}$. So

$$\rho(\Gamma_{k_1,\dots,k_l})=\sum_{i=1}^{l}k_i(2m-k_i).$$

Then, by Proposition 4.3, we get that

$$\min \deg_q \langle \Gamma_{k_1,\dots,k_l} \rangle_N \geq -bm(N-m) - \sum_{i=1}^{l} k_i(2m-k_i), \tag{4.6}$$

$$\max \deg_q \langle \Gamma_{k_1,\dots,k_l} \rangle_N \leq bm(N-m) + \sum_{i=1}^{l} k_i(2m-k_i). \tag{4.7}$$

By (2.7) and (2.8), we know that

$$\mathsf{s}_{q,N}(B^{(m)};\Gamma_{k_1,\dots,k_l}) = wm(N-m) + \sum_{i=1}^{l} \varepsilon_i k_i. \tag{4.8}$$

Since the graded dimension of $H_N(\Gamma_{k_1,\dots,k_l})$ is $\langle \Gamma_{k_1,\dots,k_l} \rangle_N$, using (4.6), (4.7) and (4.8), we get

$$\begin{aligned} & \min \deg_q H_N(\Gamma_{k_1,\dots,k_l})\|\mathsf{s}_{h,N}(B^{(m)};\Gamma_{k_1,\dots,k_l})\|\{q^{\mathsf{s}_{q,N}(B^{(m)};\Gamma_{k_1,\dots,k_l})}\} \\ \geq & \ (w-b)m(N-m) - \sum_{i=1}^{l} k_i(2m-k_i-\varepsilon_i) \\ \geq & \ (w-b)m(N-m) - lm^2 + wm \end{aligned}$$

and

$$\begin{aligned} & \max \deg_q H_N(\Gamma_{k_1,\dots,k_l})\|\mathsf{s}_{h,N}(B^{(m)};\Gamma_{k_1,\dots,k_l})\|\{q^{\mathsf{s}_{q,N}(B^{(m)};\Gamma_{k_1,\dots,k_l})}\} \\ \leq & \ (w+b)m(N-m) + \sum_{i=1}^{l} k_i(2m-k_i+\varepsilon_i) \\ \leq & \ (w+b)m(N-m) + lm^2 + wm, \end{aligned}$$

where we also used the fact that, for any integer k_i,

$$\begin{aligned} -k_i(2m-k-\varepsilon_i) & \geq -m^2 + \varepsilon_i m, \\ k_i(2m-k_i+\varepsilon_i) & \leq m^2 + \varepsilon_i m. \end{aligned}$$

By Theorem 2.6, this implies Proposition 1.1. □

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